Propagation of localized longitudinal strain waves in a plate in the presence of cubic nonlinearity

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Possible propagating longitudinal strain solitary waves in a plate are shown to be seriously altered when physical cubic nonlinearity is taken into account in the modeling. This also affects an amplification of the wave due to the transverse instability of plane-localized waves and due to the plane-wave interaction.

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I. INTRODUCTION

Recently we have improved the model equation for longitudinal strain wave propagation in a cylindrical rod by adding cubic nonlinearity [1]. While quadratic nonlinearity turns out to be responsible for a balance between nonlinearity and dispersion required for the wave localization, cubic nonlinearity also plays an important role. It makes the solitary wave wider and provides faster solitary-wave generation from an arbitrary input. This provides better agreement with experimental data. Also an increase in amplitude of the generated solitary waves or an amplification of the wave happens due to the presence of cubic nonlinearity in the model equation as demonstrated in Ref. [1]. Amplification of localized strain waves is of great importance since exceeding of some threshold value transforms elastic deformations to plastic ones. The amplitude and velocity of the localized strain wave are expressed through elastic moduli, which allow us to estimate them measuring the parameters of the wave.

To some extent, propagation of longitudinal strain waves in a plate is similar to that in a rod. Moreover, experiments regarding localized strain-wave propagation in a plate [2] have established the same difference in the values of the theoretical and experimental widths of the localized wave. Recent achievements in a nonlinear description of laterally free surface plates may be found in Refs. [3–8]. The influence of a prestress is studied in [9] and effects caused by moving defects are considered in [10], while the influence of an external medium is studied in [11]. Only quadratic nonlinearity is usually taken into account in these papers when longitudinal waves are considered. However, the coefficients at nonlinear and dispersive terms in the governing equations differ from those of equations for the rod.

Moreover, wave motion in a rod is governed by a one-dimensional (1D) nonlinear partial differential equation. Then localization may happen only along the direction of the wave propagation or along the axis of the rod. The governing equations for a plate are 2D, and two kinds of wave localization are possible. The first kind occurs along the direction of a plane-wave propagation, while the second kind happens in all directions in the plane where a wave propagates. Therefore one can anticipate that cubic nonlinearity affects 2D localization of the wave as happens for waves in fluids [12].

In this paper, we study the influence of the addition of cubic nonlinearity in the model equations on longitudinal strain wave propagation and amplification in a laterally free surface plate. The governing 2D equations are obtained in Sec. III to account for the strain evolution in the case of weakly transverse variations. A plane solitary wave solution is obtained in Sec. IV while its generation from an arbitrary input is studied in Sec. V. Transverse instability of a plane-localized wave is studied in Sec. VI as a first reason for the wave amplification. Another mechanism based on the resonant interaction between plane waves or the waves with curved fronts is considered in Sec. VII.

II. STATEMENT OF THE PROBLEM

Let us consider an isotropic elastic plate that occupies the region $-\infty<x<\infty$, $-\infty<y<\infty$, $-h<z<h$, in Cartesian coordinates (x,y,z). Assume the displacement vector in the plate is $\vec{V}=(u,v,w)$. The strain field in the reference configuration is defined by the Cauchy-Green deformation tensor $\mathbf{C}$,

$$\mathbf{C}=[\vec{\nabla}\vec{V}+(\vec{\nabla}\vec{V})^T+\vec{\nabla}\vec{V}\cdot(\vec{\nabla}\vec{V})^T]/2$$

[written in terms of a vector gradient $\vec{\nabla}\vec{V}$ and its transpose $(\vec{\nabla}\vec{V})^T$].

We choose the Murnaghan model [13] to account for a potential strain energy density in the plate given by

$$\Pi=\frac{\lambda+2\mu}{2}I_1^2-2\mu I_2+\frac{l+2m}{3}I_1^3-2mI_1I_2+nI_3+\nu_1 I_1^4+\nu_2 I_1^2 I_2+\nu_3 I_1 I_3+\nu_4 I_2^2, \tag{1}$$

where $I_k, k=1,2,3$, are the invariants of tensor $\mathbf{C}$—i.e.,

$$I_1(\mathbf{C})=C_{xx}+C_{yy}+C_{zz},$$

$$I_2(\mathbf{C})=C_{xx}C_{yy}+C_{xx}C_{zz}+C_{yy}C_{zz}-C_{xy}^2-C_{xz}^2-C_{yz}^2,$$

$$I_3(\mathbf{C})=\det\mathbf{C}.$$

Other notations introduced are the Lamé coefficients (λ,μ) and the third-order elastic moduli, or the Murnaghan moduli (l,m,n). Both the third-order elastic moduli, or the

Murnaghan moduli (l,m,n), and the fourth-order moduli $(\nu_1,\nu_2,\nu_3,\nu_4)$ can be either positive or negative [17–19].

For the kinetic energy density K we have

$$K=\frac{\rho_0}{2}\left[\left(\frac{\partial u}{\partial t}\right)^2+\left(\frac{\partial v}{\partial t}\right)^2+\left(\frac{\partial w}{\partial t}\right)^2\right], \tag{2}$$

where ρ_0 is the plate material density at time $t=t_0$.

Once the reference configuration is defined we use Hamilton's principle to obtain the governing equations setting to zero the variation of the action functional,

$$\delta\int_{t_0}^{t_1} dt\left[\int_{-h}^{h} dz\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathcal{L}dxdy\right]=0, \tag{3}$$

where $\mathcal{L}$ is the Lagrangian density per unit volume, $\mathcal{L}=K-\Pi$, with Π defined by Eq. (1). The integration in brackets in Eq. (3) is carried out at the initial time $t=t_0$. Initially, the plate is supposed to be in its natural, equilibrium state.

We impose zero-boundary conditions for the stress components $P_{zz}=0$, $P_{zx}=0$, and $P_{zy}=0$ at the surfaces of the plate, $z=\pm h$. The components P_{zz} and P_{zx} of the Piola-Kirchhoff stress tensor $\mathbf{P}$ are defined in the framework of the nine-constant Murnaghan model as

$$\begin{aligned}
P_{zz}&=(\lambda+2\mu)w_z+\lambda(v_y+u_x)+\frac{3\lambda+6\mu+2l+4m}{2}w_z^2+\frac{\lambda+2\mu+m}{2}(u_z^2+v_z^2+w_x^2+w_y^2)+(2l-2m+n)u_xv_y+\frac{\lambda+2l}{2}(u_x^2+v_y^2\\
&+2u_xw_z+2v_yw_z)+\frac{2(\lambda+m)-n}{4}(u_y^2+v_x^2)+(\mu+m)(u_zw_x+v_zw_y)+\frac{2m-n}{2}u_yv_x+\frac{\lambda+2\mu+4l+8m+8\nu_1}{2}w_z^3\\
&+\frac{\lambda+2\mu+2l+5m-\nu_2}{2}w_z(u_z^2+v_z^2+w_x^2+w_y^2)+(1+4\nu_1+\nu_2)(u_x^3+v_y^3)+\frac{2m-n-2(\nu_2+\nu_3)}{2}u_yv_xw_z+\frac{4m+\nu_3}{4}[u_yu_zv_z\\
&+(u_y+v_x)w_xw_y+u_zv_xv_z]+(3m-\nu_2)(u_zw_xw_z+v_zw_yw_z)+\frac{n+4\nu_3}{4}(u_zv_xw_y+u_yv_zw_x)+\frac{4(\mu+m)+n+\nu_3}{4}(u_yu_zw_y+v_xv_zw_x)\\
&+\frac{2\lambda+4l+2m-n-2(\nu_2+\nu_3)}{4}(u_y^2w_z+v_x^2w_z)+3(l+4\nu_1+\nu_2)w_z^2(u_x+v_y)+(\mu+2m-\nu_2-\nu_4)(u_xu_zw_x+v_yv_zw_y)\\
&+\frac{2m-n-2\nu_2-\nu_3-2\nu_4}{2}[u_yv_x(v_y+u_x)+u_xv_zw_y+u_zv_yw_x]+\frac{2(l+m)-\nu_2-\nu_4}{2}[u_x(u_z^2+w_x^2)+v_y(v_z^2+w_y^2)]\\
&+\frac{4l-2\nu_2-\nu_3-2\nu_4}{4}[u_x(u_y^2+v_x^2+v_z^2+w_y^2)+v_y(u_y^2+u_z^2+v_x^2+w_x^2)]+(l-m+0.5n+12\nu_1+5\nu_2+\nu_3+2\nu_4)u_xv_y(u_x+v_y)\\
&+(0.5\lambda+2l+12\nu_1+4\nu_2+2\nu_4)w_z(u_x^2+v_y^2)+(2l-2m+n+24\nu_1+10\nu_2+2\nu_3+4\nu_4)u_xv_yw_z,
\end{aligned} \tag{4}$$

$$\begin{aligned}
P_{zx}&=\mu(u_z+w_x)+(\lambda+2\mu+m)(u_zw_z+u_xu_z)+\frac{n}{4}v_xw_y+\frac{2m-n}{2}v_yw_x+\frac{2\lambda+2m-n}{2}u_zv_y+(\mu+m)w_x(u_x+w_z)+\frac{4\mu+n}{4}(u_yv_z\\
&+u_yw_y+v_xv_z)+\frac{3m-\nu_2}{2}w_x(u_x^2+w_z^2)+\frac{\lambda+2\mu+2l+5m-\nu_2}{2}u_z(u_x^2+w_z^2)+\frac{\lambda+2l+m-n-\nu_2-\nu_3}{2}u_zv_y^2\\
&+\frac{2(m-\nu_2-\nu_3)-n}{4}v_y^2w_x+\frac{4m+\nu_3}{4}[u_xu_y(v_z+w_y)+v_xv_z(v_y+w_z)+u_y(v_yw_y+v_zw_z)]+\frac{n+\nu_3}{4}[u_xv_xw_y+v_xw_y(v_y+w_z)]\\
&+(2l-\nu_2-0.5\nu_3-\nu_4)u_zv_y(u_x+w_z)+[2(l+m)-\nu_2-\nu_4]u_zv_yw_z+\frac{2m-n-2\nu_2-\nu_3-2\nu_4}{2}v_yw_x(u_x+w_z)+(\mu+2m-\nu_2\\
&-\nu_4)u_xw_xw_z+(m+0.5\nu_4)u_z(u_yv_x+u_zw_x+v_zw_y)+\frac{2m+\nu_4}{4}w_x(u_y^2+u_z^2+v_x^2+v_z^2+w_x^2+w_y^2)+\frac{n+2\nu_4}{4}w_x(u_yv_x+v_zw_y)\\
&+\frac{2\lambda+4m-n+\nu_4}{4}u_z(v_x^2+w_y^2)+\frac{2(\lambda+2\mu)+4m+\nu_4}{4}u_z(u_y^2+u_z^2+v_z^2+w_z^2)+\frac{2(\lambda+2\mu)+3\nu_4}{4}u_zw_x^2,
\end{aligned} \tag{5}$$

while the expression for P_{zy} may be obtained by the formal replacement in Eq. (5) of x by y, y by x, u by v, and v by u.

III. GOVERNING EQUATIONS FOR LONGITUDINAL AND SHEAR WAVES

Assume that the thickness of the plate h is much less than the typical size of the wave L, $h \ll L$, and that the amplitude B of the strain wave cannot exceed the yield point of the material. For Murnaghan's materials this means that $B \ll 1$. Then the displacement vector components may be approximated by power series in z. This allows us to use the approach developed in [20] and obtain these series so as to satisfy boundary conditions at the lateral surfaces of the plate. The series may be truncated since the nine-constant Murnaghan approximation (1) is used. Then we have

$$u = U(x,y,t) + Cz^2(U_x + V_y)_x, \tag{6}$$

$$v = V(x,y,t) + Cz^2(U_x + V_y)_y, \tag{7}$$

$$\begin{aligned} w = & -2Cz(U_x + V_y) - Dz^3\Delta(U_x + V_y) - z(q_1[U_x^2 + V_y^2] \\ & + q_2[U_y^2 + V_x^2] + q_3 U_x V_y + q_4 U_y V_x) - z(p_1[U_x^3 + V_y^3] \\ & + p_2[U_x + V_y][V_x^2 + U_y^2] + p_3 U_y V_x[V_y + U_x] \\ & + p_4 U_x V_y[U_x + V_y]), \end{aligned} \tag{8}$$

where $U(x,y,t)$ and $V(x,y,t)$ are new unknown functions, Δ is the Laplace operator,

$$C = \frac{\lambda}{2(\lambda + 2\mu)}, \quad D = \frac{\lambda^2}{6(\lambda + 2\mu)^2},$$

and we have

$$q_1 = \frac{\lambda(\lambda^2 + 3\mu\lambda + 2\mu^2) + 4l\mu^2 + 2m\lambda^2}{(\lambda + 2\mu)^3}, \quad q_2 = \frac{2\lambda + 2m - n}{4(\lambda + 2\mu)},$$

$$q_3 = \frac{\lambda^2(\lambda + 2\mu) + 8l\mu^2 + 2m(\lambda^2 - 4\lambda\mu - 4\mu^2) + n(\lambda + 2\mu)^2}{(\lambda + 2\mu)^3},$$

$$q_4 = \frac{2m - n}{2(\lambda + 2\mu)},$$

$$p_1 = \frac{\lambda^2(\lambda + \mu)}{(\lambda + 2\mu)^3} + \frac{4(\lambda + \mu)(\lambda^2 m + 2\mu^2 l)}{(\lambda + 2\mu)^4} + \frac{8(\lambda^3 m^2 - 2\mu^3 l^2 - \lambda\mu l m[\lambda - 2\mu])}{(\lambda + 2\mu)^5} + \frac{32\mu^3\nu_1}{(\lambda + 2\mu)^4} - \frac{4(\lambda - \mu)\mu\nu_2}{(\lambda + 2\mu)^3} - \frac{2\lambda\nu_4}{(\lambda + 2\mu)^2},$$

$$p_2 = \frac{\lambda^2}{2(\lambda + 2\mu)^2} + \frac{16\mu^2 l + 2\lambda(5\lambda + 2\mu)m - \lambda(\lambda + 2\mu)n}{4(\lambda + 2\mu)^3} + \frac{(\lambda m - \mu l)(2m - n)}{(\lambda + 2\mu)^3} - \frac{\mu\nu_2}{(\lambda + 2\mu)^2} + \frac{(\lambda - 2\mu)\nu_3}{4(\lambda + 2\mu)^2} - \frac{\nu_4}{2(\lambda + 2\mu)},$$

$$p_3 = \frac{2(\lambda^2 + 3\lambda\mu + 2\mu^2)m - (\lambda + \mu)n}{(\lambda + 2\mu)^3} + \frac{2(\lambda m - l)(2m - n)}{(\lambda + 2\mu)^3} - \frac{2\mu\nu_2}{(\lambda + 2\mu)^2} + \frac{(\lambda - 2\mu)\nu_3}{2(\lambda + 2\mu)^2} - \frac{\nu_4}{(\lambda + 2\mu)},$$

$$\begin{aligned} p_4 = & \frac{\lambda^2(2\lambda + \mu)}{(\lambda + 2\mu)^3} + \frac{16\mu^2(2\lambda + \mu)l + 2[5(\lambda^2(\lambda - 2\mu) - 4\mu^2(5\lambda + 2\mu)]m}{2(\lambda + 2\mu)^4} + \frac{(3\lambda + 2\mu)n}{2(\lambda + 2\mu)^2} + \frac{4\mu l[4(\mu^2 + 5\lambda\mu - \lambda^2) - (\lambda + 2\mu)^2 n]}{(\lambda + 2\mu)^5} \\ & - \frac{48\mu^3 l^2 + 4\lambda(\lambda + 2\mu)^2 mn + 16\lambda[(\lambda - \mu)^2 - 3\mu^2]m^2}{(\lambda + 2\mu)^5} + \frac{96\mu^3\nu_1}{(\lambda + 2\mu)^4} - \frac{4(2\lambda - 5\mu)\mu\nu_2}{(\lambda + 2\mu)^3} - \frac{(\lambda - 2\mu)\nu_3}{(\lambda + 2\mu)^2} - \frac{4(\lambda - \mu)\nu_4}{(\lambda + 2\mu)^2}. \end{aligned}$$

Substituting from Eqs. (6)–(8) into Eqs. (1) and (2) we get the relationships for the potential and kinetic energies in terms of U and V and their derivatives. Then the variation in the Hamilton principle (3) allows us to set to zero the expressions at variations δU and δV giving two coupled equations for U and V. In the general case these equations are lengthy and their analysis is not manageable. A simplification may be achieved assuming weak transverse variations. Then we obtain the following two coupled equations:

$$\begin{aligned} & U_{tt} - a_1 U_{xx} - a_2 U_{yy} - (a_1 - a_2)V_{xy} - a_4(U_x^2)_x - a_5(U_x^3)_x \\ & - a_6 U_{xxxx} + a_7 U_{xxtt} = 0 \end{aligned} \tag{9}$$

and

$$V_{tt} - a_1 V_{yy} - a_2 V_{xx} - (a_1 - a_2)U_{xy} = 0, \tag{10}$$

where we have set

$$a_1 = \frac{4\mu(\lambda + \mu)}{\rho_0(\lambda + 2\mu)}, \quad a_2 = \frac{\mu}{\rho_0},$$

$$a_4 = \frac{3q_1}{\rho_0}, \quad a_5 = \frac{4p_1}{\rho_0}, \quad a_6 = \frac{4h^2\lambda\mu(\lambda + \mu)}{3\rho_0(\lambda + 2\mu)^2},$$

$$a_7 = \frac{2h^2\lambda\mu}{3(\lambda + 2\mu)^2}.$$

IV. PLANE SOLITARY-WAVE SOLUTION

First, a plane-traveling solitary-wave solution of Eqs. (9) and (10) is considered when $\partial/\partial y=0$ and the dependent variables are functions of $\theta=x-ct$. Then $V=0$, and after transformation of variables, $\eta=U_\theta$, the ordinary-differential equation reduction of Eq. (9) (once integrated),

$$(c^2-a_1)\eta - a_4\eta^2 - a_5\eta^3 - (a_6-a_7c^2)\eta_{\theta\theta}=0,$$

coincides with that of the Gardner equation. Its direct integration gives rise to the solitary-wave solution

$$\eta=\frac{A}{Q\cosh(k\theta)+1}, \tag{11}$$

where

$$A=\frac{3(c^2-a_1)}{a_4}, \quad Q=\pm\sqrt{1+\frac{9a_5}{2a_4^2}(c^2-a_1)}, \tag{12}$$

$$k^2=\frac{c^2-a_1}{a_6-a_7c^2}.$$

The known exact traveling solitary-wave solution of the Gardner equation [21,22] arises with positive values of Q for sign plus in the second expression in Eqs. (12). Like in the rod [1] this solution does not result from a balance between cubic nonlinearity and dispersion. When the coefficient of cubic nonlinearity vanishes, $a_5\to 0$, then $Q\to 1$ and Eq. (11) transforms into the known solitary-wave solution of the double-dispersive equation (DDE), which was used previously to model plane waves in a plate [2,8,11]:

$$\eta=A_D\cosh^{-2}\left(\frac{1}{2}k(x-ct)\right), \tag{13}$$

where

$$A_D=\frac{3(c^2-a_1)}{2a_4}.$$

Due to Eqs. (12), k should be real and c^2 lies inside the interval

$$a_1<c^2<\frac{(\lambda+2\mu)a_1}{2\mu}. \tag{14}$$

The sign of the amplitude of both solutions is defined by the sign of the coefficient a_4. One can note that both the lower and upper bounds in Eq. (14) are greater than those in the rod [1]. The length of the permitted interval in the plate, a_6/a_7-a_1, is also greater than that in the rod. An extra condition for the reality of Q in the solution (11) holds for $a_5<0$,

$$c^2<a_1-\frac{2a_4^2}{9a_5}. \tag{15}$$

This restriction also differs from that of the rod because of the difference in the values of a_4 and a_5 for the rod and for the plate. Since the amplitude of the solitary wave in the presence of cubic nonlinearity may be written as $A_G=3(c^2-a_1)/[a_4(1+Q)]$, the amplitude ratio is

$$\frac{A_D}{A_G}=\frac{1+Q}{2}$$

and cubic nonlinearity gives rise to an increase in amplitude if $a_5<0$. Only one set of fourth-order moduli for aluminum is known to us by now from [19]. Using these data, $l=-2.9\times10^{11}$ N/m^2, $m=-3.1\times10^{11}$ N/m^2, $n=-2.3\times10^{11}$ N/m^2, $\nu_1=-1.4\times10^{12}$ N/m^2, $\nu_2=-5.3\times10^{12}$ N/m^2, $\nu_3=1.7\times10^{12}$ N/m^2, and $\nu_4=-2.9\times10^{12}$ N/m^2, one obtains $a_5<0$ and $A_G>A_D$. The opposite result has been obtained for the aluminum rod [1] where the cubic term coefficient is positive and the cubic nonlinearity provides a decrease in amplitude of the solitary wave. At the same time both a_4 and the corresponding coefficient at the quadratic nonlinear term in the equation for the rod [1] are of the same negative sign. Similarly, the quadratic term coefficients a_4 are of the same sign for polystyrene, plexiglas, copper, molibdenium, and some other materials.

A distinction in the sign of the cubic term coefficients yields a different time of formation of the solitary-waves from an arbitrary input as well as a difference in the solitary wave amplitudes. Indeed, the governing equations for the plane wave in a plate and that of the rod are so similar as to allow us to use numerical solutions found for the rod in [1]. In particular, we obtain for aluminum that solitary waves in the plate are generated slower than in the rod and they have a smaller amplitude. Moreover, a breakdown of the initial wave may happen in the plate if the amplitude of the input leads to $-2a_4/(3a_5)$, which follows from Eqs. (12). The last scenario is not realized in the aluminum rod.

Contrary to the waves in fluids the cubic term coefficient a_5 may be positive now, which makes possible a bounded solution following from Eq. (11) at negative values of Q, $Q<-1$,

$$\eta=-\frac{A}{|Q|\cosh(k\theta)-1}. \tag{16}$$

Again the sign of the amplitude is defined by the quadratic term coefficient. However, now the sign of the amplitude of the solution (16) is opposite to the sign of a_4. Hence both compression and tensile solitary waves exist for positive a_5 independent of the sign of the coefficient of the quadratic term. In particular, Eq. (11) accounts for the tensile wave for $a_4>0$ while Eq. (16) describes the compression wave. This may be important to explain recent experiments on the compression solitary-wave generation in a Plexiglas rod and in a plate [14]. Indeed, the measurements of the moduli for the Plexiglas give rise to the opposite predictions regarding the possible kind of the solitary wave. Thus, according to [15] the third-order moduli are $l=-1.09\times10^{10}$ N/m^2, $m=0.24\times10^{10}$ N/m^2, and $n=1.88\times10^{10}$ N/m^2, and the quadratic term coefficients both for the rod and for the plate are positive. This means that only a tensile wave is possible according to the theory using only quadratic nonlinearity. Another set of moduli found in [16], $l=-1.1\times10^{10}$ N/m^2, $m=-1.4\times10^{10}$ N/m^2, and $n=-1.4\times10^{10}$ N/m^2 yields a negative coefficient for a rod but positive for a plate. Certainly observation of compression waves both in the rod and in the plate may be explained using our theory; however, definite conclusions may be reached after fourth-order moduli for Plexiglas becomes available.

The solution (16) transforms in the limit $a_5 \to 0$ to the singular solution of the DDE,

$$\eta = -A_D \sinh^{-2}\left(\frac{1}{2}k(x-ct)\right). \tag{17}$$

Finally note that a plane solitary-wave solution provides the displacement field

$$u = U + \frac{\lambda z^2 U_{xx}}{2(\lambda+2\mu)}, \quad v = 0,$$

$$w = -\frac{\lambda z U_x}{\lambda+2\mu} - \frac{\lambda^2 z^3 U_{xxx}}{6(\lambda+2\mu)^2} - zq_1 U_x^2 - zp_1 U_x^3,$$

which accounts for coupled longitudinal and shear vertical (SV) bulk localized waves propagation.

V. GENERATION OF PLANE SOLITARY WAVES FROM AN ARBITRARY INPUT

The formation of the solitary waves governed by Eq. (11) for positive sign at Q has already been studied in [1]. In particular, it was found that addition of cubic nonlinearity in the governing equation affects the time required for a formation of solitary waves while the kind of the wave, tensile or compression, is defined by the sign of the coefficient at the quadratic nonlinear term a_4. The number of generated solitary waves (11) also does not depend upon the values of the cubic nonlinear term coefficient a_5. Now attention is paid to the formation of the solitary waves (16). The most interesting case corresponds to a possible formation of tensile waves when $a_4<0, a_5>0$ or of compressive waves when $a_4>0, a_5>0$. We consider the former case using various initial pulses, Gaussian, rectangular, etc. It is found that no tensile solitary wave appears for small values of a_5 similar to the results of [1]. However, as a_5 exceeds some critical value a solitary wave arises as shown in Fig. 1. The critical value depends upon the shape of the initial condition and the value of an initial velocity of the input. A zero initial velocity yields a higher critical value since waves propagate in both directions and two solitary waves should be generated at least. That is why no solitary waves are seen in Fig. 2 for the same shape of the initial condition and the same values of the equation coefficients as used for Fig. 1. A further increase in a_5 results in an increase in the number of solitary waves. These waves propagate in opposite directions if the initial velocity is zero (see Fig. 3), while the same number of unidirectional waves propagate for nonzero initial velocity (see Fig. 4). A further increase in the initial velocity results in an increase of the number of solitary waves. The value of the quadratic nonlinear term coefficient also affects this number.

VI. PLANE-WAVE INSTABILITY

Now consider the variations in y that may cause a plane-wave instability but also always provide the appearance of shear horizontal (SH) components in the wave field. Assume thus small transverse disturbances on the plane solitary wave in the form

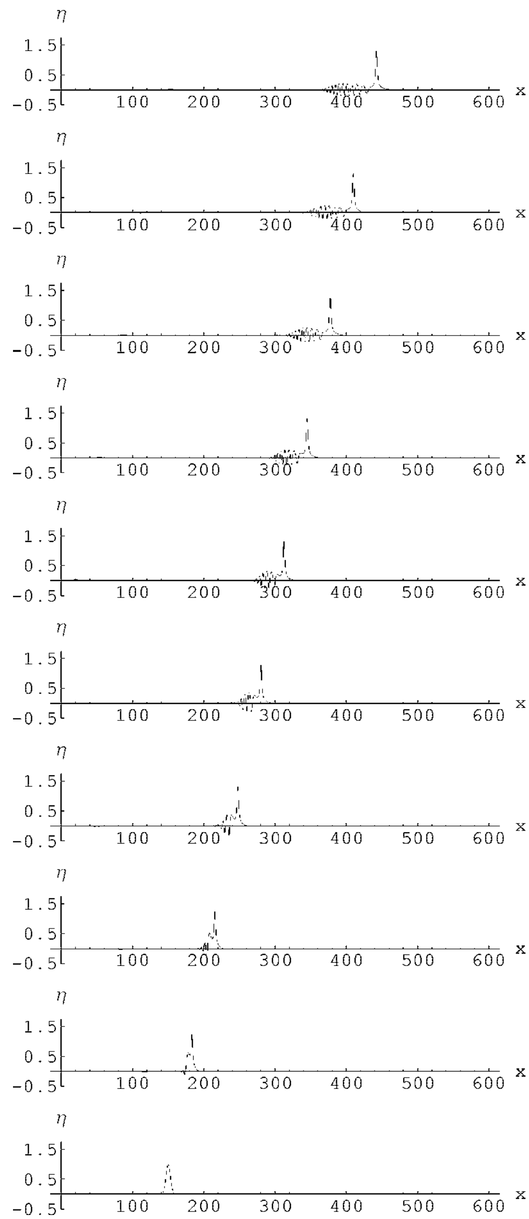

FIG. 1. Solitary strain-wave generation from initial Gaussian input for nonzero initial velocity. Propagation of solitary waves is described by the exact solution (16). Here and in the following the space x axis is given in arbitrary units of length while strain η is nondimensional.

$$U(\theta,t,y) = U_0 + \delta w_u(\theta,\tau)\exp(\kappa t + \imath s y), \tag{18}$$

$$V(\theta,t,y) = \delta w_v(\theta,t)\exp(\kappa t + \imath s y), \tag{19}$$

where $\delta \ll 1$ and $U_{0,\theta} = \eta$. Then coupled linear equations arise for the functions w_u and w_v after substituting Eqs. (18) and (19) into Eqs. (9) and (10),

$$\begin{aligned}(c^2-a_1)w_{u,\theta\theta} - 2a_4(\eta w_{u,\theta})_\theta - 3a_5(\eta^2 w_{u,\theta})_\theta \\ - (a_6 - a_7c^2)w_{u,\theta\theta\theta\theta} \\ = 2c\kappa(w_{u,\theta} + a_7\kappa w_{u,\theta\theta\theta})\ \imath(a_1-a_2)s w_{v,\theta} \\ - (a_2 s^2 + \kappa^2)w_u - \kappa^2 a_7 w_{u,\theta\theta}\end{aligned} \tag{20}$$

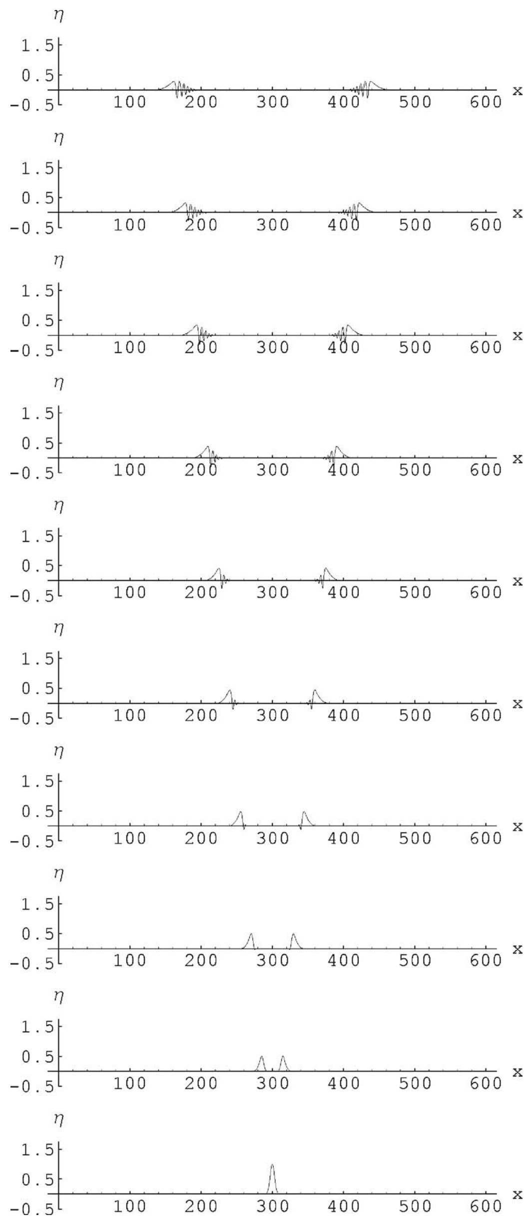

FIG. 2. No solitary-wave generation from initial Gaussian input with zero initial velocity. Dispersion gives rise to the splitting of the input into two oscillating waves pockets.

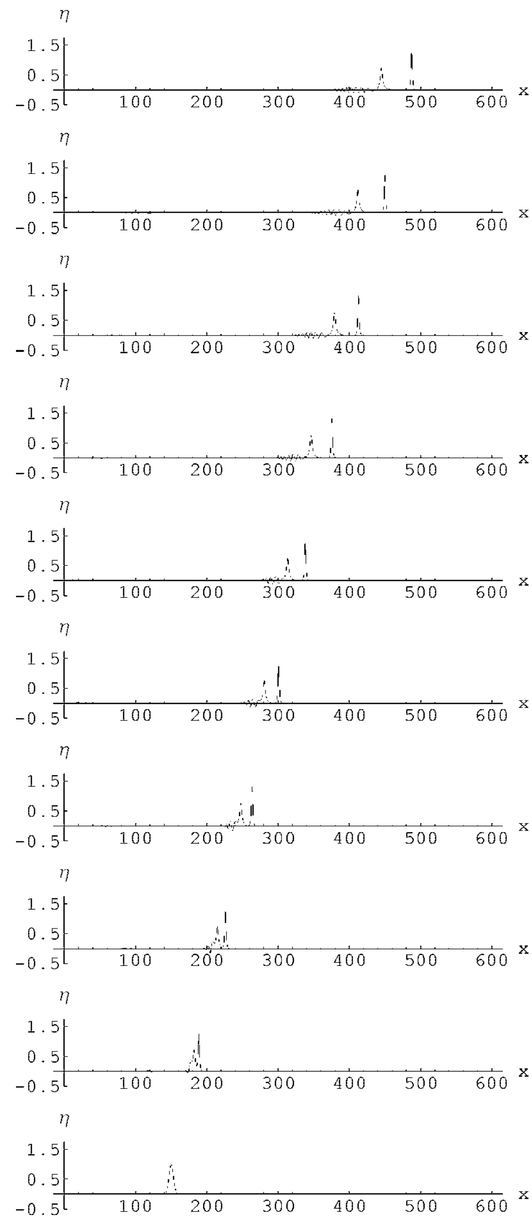

FIG. 3. Generation of two strain solitary waves from Gaussian input for nonzero initial velocity. Each of generated solitary waves is described by the exact solution (16).

and

$$(c^2 - a_2)w_{v,\theta\theta} = 2c\kappa w_{v,\theta} - \imath(a_1 - a_2)sw_{u,\theta} - \kappa^2 w_v - a_1 s^2 w_v. \tag{21}$$

We have to consider further simplification to obtain an analytical solution of Eqs. (20) and (21). Assume that transverse variations are weak and s is small; then, a solution is sought in the form

$$w_u = w_{u0} + sw_{u1} + s^2 w_{u2} + \cdots,$$

$$w_v = sw_{v1} + s^2 w_{v2} + \cdots, \quad \kappa = s\kappa_1 + s^2\kappa_2 + \cdots .$$

In the leading order, we have

$$L(w_{u0}) = 0,$$

where

$$L(w_{u0}) = (c^2 - a_1)w_{u0,\theta\theta} - 2a_4(\eta w_{u0,\theta})_\theta - 3a_5(\eta^2 w_{u0,\theta})_\theta - (a_6 - a_7 c^2)w_{u0,\theta\theta\theta\theta}. \tag{22}$$

Hence $w_{u0} = \eta$. At the next order, we get

$$L(w_{1u}) = 2c\kappa_1 w_{u0,\theta} + 2a_7 c\kappa_1 w_{u0,\theta\theta\theta},$$

$$(c^2 - a_2)w_{v1,\theta\theta} = -\imath(a_1 - a_2)w_{u0,\theta},$$

and the solution is

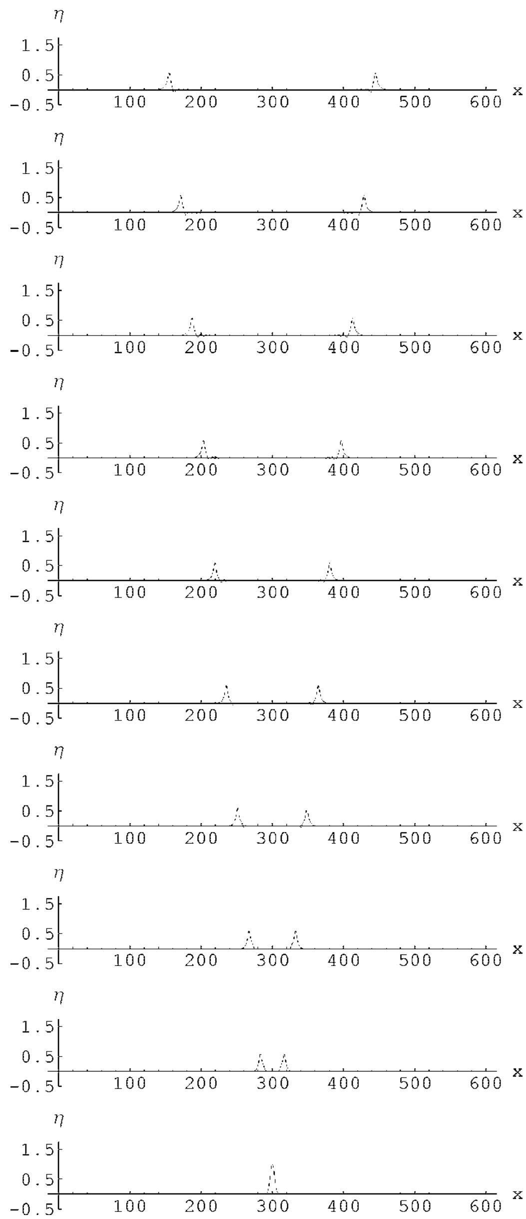

FIG. 4. Solitary-wave formation for zero initial velocity.

$$w_{u1,\theta}=\frac{3\kappa_1 c[Q^2k\theta(1+k^2a_7)\sinh(k\theta)-(1+Q^2)\cosh(k\theta)-2Q]}{a_4Q[1+Q\cosh(k\theta)]^2},$$

$$w_{v1,\theta}=\frac{\imath(a_1-a_2)\eta}{c^2-a_2}.$$

The next-order equation

$$L(w_{2u})=F,$$

where

$$F=2c\kappa_1(w_{u1,\theta}+a_7w_{u1,\theta\theta\theta})+2c\kappa_2(\eta_\theta+a_7\eta_{\theta\theta\theta}) - [a_2+\kappa_1^2+(a_1-a_2)^2/(c^2-a_1)]\eta-a_7\kappa_1^2\eta_{\theta\theta},$$

should satisfy the solvability orthogonality condition

$$\int_{-\infty}^{\infty}F\eta d\theta=0.$$

It allows us to obtain the following equation for κ_1:

$$\kappa_1^2\left(\frac{4(1-Q^2)}{a_5Q^2}-\frac{8a_4^2G_1(Q)}{9a_5^2c^2}+a_7k^2\left[\frac{8G_1(Q)}{a_5(Q^2-1)}+\frac{4a_4^2G_2(Q)}{27a_5^2}\right] - \frac{2a_7^2k^4G_2(Q)}{3a_5(1-Q^2)}\right)-\frac{8a_4^2[(a_1-a_2)^2+a_2(c^2-a_1)]G_1(Q)}{9a_5^2(c^2-a_2)} =0, \quad (23)$$

where

$$G_1(Q)=Q^2-1+2\sqrt{1-Q^2}\text{arctanh}\sqrt{\frac{1-Q}{1+Q}},$$

$$G_2(Q)=\frac{(1+2Q^2)\sqrt{1-Q^2}-6Q^2\text{arctanh}\sqrt{\frac{1-Q}{1+Q}}}{\sqrt{1-Q^2}}.$$

The sign of κ_1^2 is most important for an instability. Note that $Q>1$ for $a_5>0$ and $0<Q<1$ for $a_5<0$. Calculating limiting values of $G_1(Q)$ and $G_2(Q)$ for $Q=0,1,\infty$ and analyzing signs of their first derivatives one can check that $G_1(Q)$ and $G_2(Q)$ are always positive. Also it is possible to use a representation of arctanh through log for this analysis. Then the sign of a_5 does not affect the sign of κ_1^2. One can see that for $a_7=0$ it is always negative and that it corresponds to stability. In our case $a_7>0$, which provides a possibility for a positive sign of κ_1^2, which may be checked considering the limiting case $c^2\to a_6/a_7$. The threshold value of c (when the sign is changed) depends upon the value of the cubic nonlinear term coefficient a_5, but it also depends upon the values of the other coefficients a_4, a_6, and a_7. Positive κ_1^2 corresponds to the unstable case that provides transverse modulation of an initial plane solitary wave, and the two-dimensionally localized strain waves may exist. Often the modulation is accompanied by wave amplification [23].

For comparison, let us consider the stability of the solitary wave (13), which exists in the absence of a cubic nonlinear term in the governing equation. Using the same form of the solution, Eqs. (18) and (19), one can obtain an equation for κ_1^2 of the form

$$c^2(c^2-a_2)F\kappa_1^2+5(c^2-a_1)[a_2(c^2-a_2)+(a_1-a_2)^2]=0, \quad (24)$$

where

$$F=5(c^2-a_1+3)-a_7k^2(c^2-a_1+10)-a_7^2k^4.$$

Due to Eq. (12), we can see that permitted velocities in the absence of mixed dispersion, $a_7=0$, provide a negative sign of κ_1^2. However, positive a_7 affects the instability; F is positive for $c^2=a_1$ but negative for $c^2=a_6/a_7$.

In both the stable and unstable cases the plane longitudinal and SV waves are modulated in the transverse direction. Moreover, an SH mode appears in contrast to the case of plane-localized strain wave propagation.

VII. AMPLIFICATION DUE TO THE WAVE INTERACTION

Two-dimensional wave generation and amplification happen even in the stable case. Usually, the velocity lies near $\sqrt{a_1}$ and the governing equations (9) and (10) may be rewritten in a dimensionless form. Two cases may be considered. The first happens if the quadratic term coefficient is not small. In this case a ratio is introduced, $B=O(H^2/L^2)$, which provides a balance between nonlinearity and dispersion, and the small parameter is $\varepsilon=H/L$. Suppose that the scale Y for the variable y corresponds to the scale L for x as $Y/L=O(\varepsilon)$. Then the cubic nonlinear term in the governing equation may be neglected, the fast phase variable $\theta=x-\sqrt{a_1}t$ and the slow variable $\tau=\varepsilon^2 t$ are introduced, and the governing equation becomes the well-known Kadomtsev-Petviashvili (KP) equation [24]

$$2\sqrt{a_1}\,\eta_{\tau\theta}+a_4(\eta^2)_{\theta\theta}+(a_6-a_1a_7)\,\eta_{\theta\theta\theta\theta}=-a_1\,\eta_{yy}. \quad (25)$$

It is well known [24] that the solitary-wave solution [similar to Eq. (13)] of this equation is stable for a_1 and $a_6-a_1a_7>0$. Hence no two-dimensional localized solitary wave appears as a result of the transverse modulation of the plane solitary wave. However, it was found in [25] that even initially noninteracting semiplane solitary waves give rise to the formation of a 2D localized wave structure whose amplitude may be almost 4 times higher than the amplitude of the initial waves. The amplification of the strain wave in the plate depends upon the angle between initial waves: there exists a critical value of it when the localized wave achieves its maximum. Moreover, if initial waves with curved fronts are chosen, the amplitude of the localized wave may be up to 14 times higher depending upon the curvature of the incident waves.

When the quadratic and cubic nonlinearities in Eq. (9) are of the same order the 2D Gardner equation arises instead of the KP equation,

$$2\sqrt{a_1}\,\eta_{\tau\theta}+a_4(\eta^2)_{\theta\theta}+a_5(\eta^3)_{\theta\theta}+(a_6-a_1a_7)\,\eta_{\theta\theta\theta\theta}=-a_1\,\eta_{yy}. \quad (26)$$

Again a plane solitary-wave solution is stable against transverse disturbances. Numerical analysis of the semiplane wave interaction in the framework of Eq. (26) has been done in [12]. It was found that formation of the 2D localized wave happens similar to the KP equation case. However, cubic nonlinearity provides both quantitative and qualitative distinctions. In particular, there is no critical angle between the incident waves where the maximum amplitude of the 2D wave is realized.

VIII. CONCLUSIONS

The equations we obtained to account for strain-wave propagation in a plate allow one to explain deviations in the width of the waves observed in experiments and described by the double-dispersive equation. The most important is the solitary-wave solution (16), which gives rise to the existence of both compression and tensile waves for materials with suitable fourth-order moduli. Moreover, the formation of these waves from an arbitrary input differs from that of the known solution (11). To some extent, the last solution is an extension of the solitary-wave solution of the double-dispersion equation while the solution (16) is intrinsic in Eq. (9).

Also the addition of cubic nonlinearity in the governing equation defines the form of the leading-order solitary-wave solution yielding a mutual influence on the stability by mixed dispersion and cubic nonlinearity. In the absence of cubic nonlinearity the coefficients of nonlinear terms play no role in it. It is easy to see in Eq. (24) that the value of the coefficient of the quadratic nonlinear term a_4 does not affect the instability.

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